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ORIGINAL ARTICLE

Contiguous relations for ${}_2F_1$ hypergeometric series

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Abstract Two Gauss functions are said to be contiguous if they are alike except for one pair of parameters, and these differ by unity. Contiguous relations are of great use in extending numerical tables of the function. In this paper we will introduce a new method for computing such types of relations.

1. Introduction

The study of hypergeometric series was launched many years ago by Euler, Gauss and Riemann; such series are the subject of considerable research. Hypergeometric series have a somewhat formidable notation, which takes a little time to get used to.

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss, 1813) in which he considered the infinite series

$$1+\frac{a_1a_2}{1\cdot a_3}z+\frac{a_1(a_1+1)a_2(a_2+1)}{1\cdot 2\cdot a_3(a_3+1)}z^2 +\frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)}{1\cdot 2\cdot 3\cdot a_3(a_3+1)(a_3+2)}z^3+\cdots \tag{1}$$

as a function of a_1, a_2, a_3, z, where it is assumed that $a_3 \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $Re(a_3 - a_1 - a_2) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula

$$F(a_1,a_2;a_3;1)=\frac{\Gamma(a_3)\Gamma(a_3-a_1-a_2)}{\Gamma(a_3-a_1)\Gamma(a_3-a_2)},\quad Re(a_3-a_1-a_2)>0 \tag{2}$$

for the sum of his series when $z = 1$ and $Re(a_3 - a_1 - a_2) > 0$.

Although Gauss used the notation $F(a_1, a_2, a_3, z)$ for his series, it is now customary to use $F[a_1, a_2; a_3; z]$ or either of the notations

$${}_2F_1(a_1,a_2;a_3;z),\quad {}_2F_1\left[\begin{matrix}a_1,a_2\\ a_3\end{matrix};z\right] \tag{3}$$

for the series (and for its sum when it converges), because these notations separate the numerator parameters a_1, a_2 from the denominator parameter a_3 and the variable z. In view of Gauss' paper, his series is frequently called Gauss' series. However, since the special case $a_1 = 1$, $a_2 = a_3$ yields the geometric series

$$1+z+z^2+z^3+\cdots \tag{4}$$

Gauss' series is also called (ordinary) hypergeometric series or the Gauss hypergeometric series. For more details about hypergeometric series and their contiguous relations, see [1–4].

Two hypergeometric functions with the same argument z are *contiguous* if their parameters a_1, a_2 and a_3 differ by integers. Gauss derived analogous relations between ${}_2F_1[a_1, a_2; a_3; z]$

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and any two contiguous hypergeometrics in which a parameter has been changed by ± 1. Rainville [5] generalized this to cases with more parameters.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series, they can be used to evaluate a hypergeometric function that is contiguous to a hypergeometric series which can be satisfactorily evaluated. Contiguous relations are also used to make a correspondence between Lie algebras and special functions. The correspondence yields formulas of special functions [6].

The 15 Gauss contiguous relations for ${}_2F_1[a_1, a_2; a_3; z]$ hypergeometric series imply that any three ${}_2F_1[a_1, a_2; a_3; z]$ series whose corresponding parameters differ by integers are linearly related (over the field of rational functions in the parameters). In [7], several properties of coefficients of these general contiguous relations were proved and then used to propose effective ways to compute contiguous relations. In [8], contiguous relations were used to establish and prove sharp inequalities between the Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean. More details about contiguous relations and their application can be found in [9–14].

In this paper, we will extend the results obtained in [15], to prove different identities that relate between the contiguous functions of ${}_2F_1[a_1, a_2; a_3; z]$ hypergeometric functions. We will generalize the method of Theorem 1.1. of Vidúnas in [7], in which he summarizes some properties of the coefficients of contiguous relations. This method will be useful in computations and application of contiguous relations.

The paper is organized as follows: In Section 2, we introduce our method of computations; in Section 3 we introduce our main theorem in which we generalize the operators we defined in Section 2, while in Section 4, we use *Mathematica* to show how helpful is our main theorem in deriving contiguous function relations as well as to obtain any of their consequences.

2. Computations

Gauss defined as *contiguous* to ${}_2F_1[a_1, a_2; a_3; z]$ each of the six functions obtained by increasing or decreasing one of the parameters by unity [16, pp. 555–566]. Thus ${}_2F_1[a_1, a_2; a_3; z]$ is contiguous to the six functions

$$ {}_2F_1[a_1 \pm 1, a_2; a_3; z], \quad {}_2F_1[a_1, a_2 \pm 1; a_3; z] \quad \text{and} \quad {}_2F_1[a_1, a_2; a_3 \pm 1; z] $$

Gauss proved that between ${}_2F_1[a_1, a_2; a_3; z]$ and any two of its contiguous functions, there exists a linear relation with coefficients at most linear in z. These relationships are of great use in extending numerical tables of the function, since for one fixed value of z, it is necessary only to calculate the values of the function over two units in a_1, a_2 and a_3, and apply some recurrence relations in order to find the function values over a large range of values of a_1, a_2 and a_3 in this particular z-plane. A contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations.

In this section, we will introduce our method of computations from which we will be able to prove any type of contiguous relation, and for simplicity in the notation, let us introduce the following definition:

Definition 1. Let $\mathcal{A}_i^{\alpha_i} : X \to X$, $(i = 1, 2, 3)$, where X is the set of all Gauss' functions ${}_2F_1[a_1, a_2; a_3; z]$ with variable z, and parameters a_1, a_2 and a_3 such that $a_3 \neq 0, -1, -2, \ldots$, then

$$ \mathcal{A}_1^{\alpha_1}(C[a_1, a_2, a_3]{}_2F_1[a_1, a_2; a_3; z]) = C[a_1 + \alpha_1, a_2, a_3]\ {}_2F_1[a_1 + \alpha_1, a_2; a_3; z] \tag{5} $$

$$ \mathcal{A}_2^{\alpha_2}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = C[a_1, a_2 + \alpha_2, a_3]\ {}_2F_1[a_1, a_2 + \alpha_2; a_3; z] \tag{6} $$

$$ \mathcal{A}_3^{\alpha_3}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = C[a_1, a_2, a_3 + \alpha_3]{}_2F_1[a_1, a_2; a_3 + \alpha_3; z] \tag{7} $$

where α_i, $i = 1, 2, 3$ are any integers, and $C[a_1, a_2, a_3]$ is an arbitrary constant function of a_1, a_2 and a_3 such that for any such operators

$$ \mathcal{A}_i^{\alpha_i}\mathcal{A}_i^{-\alpha_i}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = \mathcal{I}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) $$

and $\mathcal{I}$ is the identity operator defined on X with

$$ \mathcal{I}^k(C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z]) = \mathcal{I}(C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z]) = C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z];\ \forall F \in X $$

We have the following theorem:

Theorem 2. *Let $\mathcal{A}_i^{\alpha_i}, i = 1,2,3$ and $\mathcal{I}$ defined as in Definition (1), then*

$$ \mathcal{A}_3^{-1} = \frac{a_1}{a_3 - 1}\mathcal{A}_1 + \frac{a_3 - a_1 - 1}{a_3 - 1}\mathcal{I}; \quad a_3 \neq 1 \tag{8} $$

$$ \mathcal{A}_2^{-1} = \frac{a_1(z - 1)}{a_2 - a_3}\mathcal{A}_1 + \frac{a_1 + a_2 - a_3}{a_2 - a_3}\mathcal{I}; \quad a_2 \neq a_3 \tag{9} $$

$$ \mathcal{A}_1^{-1} = \frac{a_1(z - 1)}{a_1 - a_3}\mathcal{A}_1 + \frac{2a_1 + (a_2 - a_1)z - a_3}{a_1 - a_3}\mathcal{I}; \quad a_1 \neq a_3 \tag{10} $$

$$ \mathcal{A}_2 = \frac{a_1}{a_2}\mathcal{A}_1 + \frac{a_2 - a_1}{a_2}\mathcal{I}; \quad a_2 \neq 0 \tag{11} $$

$$ \mathcal{A}_3 = \frac{a_1a_3(z - 1)}{(a_1 - a_3)(a_3 - a_2)z}\mathcal{A}_1 - \frac{a_3((a_3 - a_2)z - a_1)}{(a_1 - a_3)(a_3 - a_2)z}\mathcal{I}; \quad a_1 \neq a_3, \ \ a_2 \neq a_3 \text{ and } z \neq 0 \tag{12} $$

Proof 1. To prove (8), from Eq. (45) of [15], and with $\alpha_1 = \alpha_2 = \alpha_3 = 0$, one has

$$ \left[\mathcal{I} - \mathcal{A}_1^{-1} - \frac{a_2}{a_3}z\mathcal{A}_2\mathcal{A}_3\right]\ {}_2F_1[a_1, a_2; a_3; z] = 0 $$

that is

$$ \mathcal{A}_1 = \mathcal{I} + \frac{a_2}{a_3}z\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 \tag{13} $$

Now using (47) of [15], and with $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we will have

$$\mathcal{I} - \mathcal{A}_3 - \frac{a_1 a_2}{a_3(a_3+1)} z\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^2 = 0$$

applying $\mathcal{A}_3^{-1}$ to both sides that is $a_3 \to a_3 - 1$, we will have

$$\mathcal{A}_3^{-1} = \mathcal{I} + \frac{a_1 a_2}{a_3(a_3-1)} z\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 \tag{14}$$

Eliminating $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3$ from (13) and (14), one gets (8).

Now using symmetry on (13), [Remark 1 – Section 3] in [15] we will have

$$\mathcal{A}_2 = \mathcal{I} + \frac{a_1}{a_3} z\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 \tag{15}$$

from which solving both (13) and (15) by eliminating $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3$, formula (11) holds.

Moreover, applying $\mathcal{A}_2\mathcal{A}_3$ on both sides of (13), that is ($a_2 \to a_2 + 1$ and $a_3 \to a_3 + 1$), then

$$\mathcal{A}_2 = \frac{a_1}{a_3}\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 + \frac{a_3 - a_1}{a_3}\mathcal{A}_2\mathcal{A}_3 \tag{16}$$

and from (13)

$$\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 = \frac{a_3}{a_2 z}(\mathcal{A}_1 - \mathcal{I}) \tag{17}$$

or equivalently

$$\mathcal{A}_2\mathcal{A}_3 = \frac{a_3}{a_2 z}(\mathcal{I} - \mathcal{A}_1^{-1}) \tag{18}$$

Now, using (11), (17) and (18), formula (10) holds.

By the same method, formulas (9) and (12) can be hold. □

Although Gauss relations can be proved by the expansion of the various power series in z, and equating the coefficients of z^n throughout, rewriting these relations in their corresponding operator forms makes their proofs simplified by using Theorem (2).

Theorem (2), can be of a great help in proving several types of contiguous relations such as:

1. All the 15 Gauss contiguous relations.
2. Functional identities in which relations between contiguous functions are given, [16].
3. Recurrence identities, with consecutive neighbors, in which one parameter in one of its contiguous function is shifted by ± 1, while one parameter in one of its other contiguous functions is shifted by ± 2 (07.23.17.0001.01–07.23.17.0004.01) [17].

Example 3. To prove the Gauss relation

$$(a_1 + a_2 - a_3)\ {}_2F_1[a_1, a_2; a_3; z] - a_1(1-z)\ {}_2F_1[a_1+1, a_2; a_3; z] + (a_3 - a_2)\ {}_2F_1[a_1, a_2 - 1; a_3; z] = 0$$

which can be rewritten in operator form as

$$[(a_1 + a_2 - a_3)\mathcal{I} - a_1(1-z)\mathcal{A}_1 + (a_3 - a_2)\mathcal{A}_2^{-1}]\ {}_2F_1 = 0$$

Using formula (9) of Theorem (2), we get

$$\text{L.H.S.} = \left[(a_1 + a_2 - a_3)\mathcal{I} - a_1(1-z)\mathcal{A}_1 + (a_3 - a_2) \times \left(\frac{a_1(z-1)}{a_2 - a_3}\mathcal{A}_1 + \frac{a_1 + a_2 - a_3}{a_2 - a_3}\mathcal{I}\right)\right]\ {}_2F_1 = 0 = \text{R.H.S.}$$

3. General forms

In this section we give general forms for the operators $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ given in Definition (1). In other words, we will reduce any shift of the forms ${}_2F_1[a_1 + n, a_2; a_3; z]$, ${}_2F_1[a_1, a_2 + n; a_3; z]$, and ${}_2F_1[a_1, a_2; a_3 + n; z]$ to just ${}_2F_1[a_1 + 1, a_2; a_3; z]$, that is we will reduce all shifts of the form $\mathcal{A}_1^n\ {}_2F_1$, $\mathcal{A}_2^n\ {}_2F_1$ and $\mathcal{A}_3^n\ {}_2F_1$ to the form $\mathcal{A}_1\ {}_2F_1$, which enables us to prove any contiguous relation having more than one shifted parameter.

In the following example, we express $\mathcal{A}_1^2$, $\mathcal{A}_2^2$ and $\mathcal{A}_3^2$ in terms of operator $\mathcal{A}_1$.

Example 4. To express $\mathcal{A}_1^2$, $\mathcal{A}_2^2$ and $\mathcal{A}_3^2$ in terms of operator $\mathcal{A}_1$, from (10), we have

$$\mathcal{A}_1 = \frac{a_3 + (a_1 - a_2)z - 2a_1}{a_1(z-1)}\mathcal{I} + \frac{a_1 - a_2}{a_1(z-1)}\mathcal{A}_1^{-1}$$

applying $\mathcal{A}_1$ on both sides, one gets

$$\mathcal{A}_1^2 = \frac{a_3 + (a_1 - a_2 + 1)z - 2(a_1+1)}{(a_1+1)(z-1)}\mathcal{A}_1 + \frac{a_1 - a_3 + 1}{(a_1+1)(z-1)}\mathcal{I} \tag{19}$$

Solving both (9) and (11) for $\mathcal{A}_2$, we will have

$$\mathcal{A}_2 = \frac{a_2 - a_3}{a_2(z-1)}\mathcal{A}_2^{-1} + \frac{(a_2 - a_1)(z-1) - (a_1 + a_2 - a_3)}{a_2(z-1)}\mathcal{I}$$

from which by applying $\mathcal{A}_2$ on both sides, one can easily obtain

$$\mathcal{A}_2^2 = \frac{a_3 + (a_2 + 1 - a_1)z - 2(a_2+1)}{(a_2+1)(z-1)}\mathcal{A}_2 + \frac{a_2 + 1 - a_3}{(a_2+1)(z-1)}\mathcal{I} \tag{20}$$

Now using (11) to eliminate $\mathcal{A}_2$, we get

$$\mathcal{A}_2^2 = \frac{a_3 + (a_2 + 1 - a_1)z - 2(a_2+1)}{(a_2+1)(z-1)}\left[\frac{a_1}{a_2}\mathcal{A}_1 + \frac{a_2 - a_1}{a_2}\mathcal{I}\right] + \frac{a_2 + 1 - a_3}{(a_2+1)(z-1)}\mathcal{I}$$

that is

$$\mathcal{A}_2^2 = \frac{a_1[(a_3 - 2a_2 - 2) + (a_2 - a_1 + 1)z]}{a_2(a_2+1)(z-1)}\mathcal{A}_1 + \frac{(a_2 - a_1)[(a_3 - 2a_2 - 2) + (a_2 - a_1 + 1)z] + a_2(a_2 - a_3 + 1)}{a_2(a_2+1)(z-1)}\mathcal{I}$$

Also from (8) and (12) eliminating $\mathcal{A}_1$, we get

$$\mathcal{A}_3 = \frac{a_3[a_3 - 1 + (a_1 + a_2 - 2a_3 + 1)z]}{(a_3 - a_2)(a_1 - a_3)z}\mathcal{I} + \frac{a_3(a_3 - 1)(z-1)}{(a_3 - a_2)(a_1 - a_3)z}\mathcal{A}_3^{-1}$$

applying $\mathcal{A}_3$ to both sides, then

$$\mathcal{A}_3^2 = \frac{(a_3+1)[a_3+(a_1+a_2-2a_3-1)z]}{(a_3-a_2+1)(a_1-a_3-1)z}\mathcal{A}_3 + \frac{a_3(a_3+1)(z-1)}{(a_3-a_2+1)(a_1-a_3-1)z}\mathcal{I} \tag{21}$$

Again from (12), we may rewrite (21) as

$$\mathcal{A}_3^2 = \frac{(a_3+1)[a_3+(a_1+a_2-2a_3-1)z]}{(a_3-a_2+1)(a_1-a_3-1)z} \times \left(\frac{a_1a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}\mathcal{A}_1 - \frac{a_3[(a_3-a_2)z-a_1]}{(a_1-a_3)(a_3-a_2)z}\mathcal{I}\right) + \frac{a_3(a_3+1)(z-1)}{(a_3-a_2+1)(a_1-a_3-1)z}\mathcal{I}$$

simplifying, we get

$$\mathcal{A}_3^2 = \frac{a_1a_3(a_3+1)(z-1)[a_3-(2a_3-a_1-a_2+1)z]}{(a_1-a_3)(a_1-a_3-1)(a_3-a_2)(a_3-a_2+1)z^2}\mathcal{A}_1 + \frac{a_3(a_3+1)[(a_1-(a_3-a_2)z)(a_3-(2a_3-a_1-a_2+1)z)+(a_1-a_3)(a_3-a_2)(z-1)z]}{(a_1-a_3)(a_1-a_3-1)(a_3-a_2)(a_3-a_2+1)z^2}\mathcal{I} \tag{22}$$

Results obtained in the last example can be easily used to verify recurrence identities of "*consecutive neighbors*", such as

$$[(a_2-a_3+1)\mathcal{I} - [2a_2-a_3+2+(a_1-a_2-1)z]\mathcal{A}_2 - (a_2+1)(z-1)\mathcal{A}_2^2]\ {}_2F_1 = 0$$

For simplicity in the notation, let us introduce the following definition:

Definition 5. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the operators in Definition (1), then we define the shifted matrix operator $\mathcal{D}_{\alpha_1,\alpha_2,\alpha_3}$ as

$$\mathcal{D}_{\alpha_1,\alpha_2,\alpha_3} = \begin{bmatrix} \mathcal{A}_1^{\alpha_1} & 0 & 0 \\ 0 & \mathcal{A}_2^{\alpha_2} & 0 \\ 0 & 0 & \mathcal{A}_3^{\alpha_3} \end{bmatrix}$$

for all $\alpha_i \in \mathbb{Z}$, $i = 1,2,3$.

In addition, and for any diagonal matrix K of order 3,

$$K = \begin{bmatrix} k_1(a_1,a_2;a_3;z) & 0 & 0 \\ 0 & k_2(a_1,a_2;a_3;z) & 0 \\ 0 & 0 & k_3(a_1,a_2;a_3;z) \end{bmatrix}$$

where $k_i(a_1, a_2; a_3; z)$, $i = 1, 2, 3$ is a function of z with constants a_1, a_2 and a_3, we will have

$$\mathcal{D}_{\alpha_1,\alpha_2,\alpha_3}K = \begin{bmatrix} \mathcal{A}_1^{\alpha_1}k_1(a_1,a_2;a_3;z) & 0 & 0 \\ 0 & \mathcal{A}_2^{\alpha_2}k_2(a_1,a_2;a_3;z) & 0 \\ 0 & 0 & \mathcal{A}_3^{\alpha_3}k_3(a_1,a_2;a_3;z) \end{bmatrix} = \begin{bmatrix} k_1(a_1+\alpha_1,a_2;a_3;z) & 0 & 0 \\ 0 & k_2(a_1,a_2+\alpha_2;a_3;z) & 0 \\ 0 & 0 & k_3(a_1,a_2;a_3+\alpha_3;z) \end{bmatrix}$$

and as a special case, if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, we use the notation

$$\mathcal{D}_\alpha := \mathcal{D}_{\alpha,\alpha,\alpha} = \begin{bmatrix} \mathcal{A}_1^{\alpha} & 0 & 0 \\ 0 & \mathcal{A}_2^{\alpha} & 0 \\ 0 & 0 & \mathcal{A}_3^{\alpha} \end{bmatrix}$$

such that

$$\mathcal{D}_\alpha X_i = \begin{bmatrix} \mathcal{A}_1^{\alpha} & 0 & 0 \\ 0 & \mathcal{A}_2^{\alpha} & 0 \\ 0 & 0 & \mathcal{A}_3^{\alpha} \end{bmatrix}\begin{bmatrix} \mathcal{A}_1^{i} \\ \mathcal{A}_2^{i} \\ \mathcal{A}_3^{i} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1^{i+\alpha} \\ \mathcal{A}_2^{i+\alpha} \\ \mathcal{A}_3^{i+\alpha} \end{bmatrix} = X_{i+\alpha}$$

and

$$\mathcal{D}_\alpha K = \begin{bmatrix} \mathcal{A}_1^{\alpha} & 0 & 0 \\ 0 & \mathcal{A}_2^{\alpha} & 0 \\ 0 & 0 & \mathcal{A}_3^{\alpha} \end{bmatrix}\begin{bmatrix} k_1(a_1,a_2;a_3;z) & 0 & 0 \\ 0 & k_2(a_1,a_2;a_3;z) & 0 \\ 0 & 0 & k_3(a_1,a_2;a_3;z) \end{bmatrix} = \begin{bmatrix} k_1(a_1+\alpha,a_2,a_3;z) & 0 & 0 \\ 0 & k_2(a_1,a_2+\alpha,a_3;z) & 0 \\ 0 & 0 & k_3(a_1,a_2,a_3+\alpha;z) \end{bmatrix} = K_\alpha$$

Finally,

$$\mathcal{D}_\alpha(KX_i) = \mathcal{D}_\alpha(K)\mathcal{D}_\alpha(X_i) = K_\alpha X_{i+\alpha}$$

The following lemma enables us to express the nth power of any shifted operator $\mathcal{A}_i$, $(i = 1,2,3)$ as a recurrence relation of $(n-1)$th and $(n-2)$th powers of such operators.

Lemma 6. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the operators defined as in Definition (1), then*

$$X_n = K_{n-1}X_{n-1} + T_{n-1}X_{n-2}; \qquad \forall n \in \mathbb{Z} \tag{24}$$

where X_n defined as in (28), and

$K_n = \mathcal{D}_nK_0$ and $T_n = \mathcal{D}_nT_0$

where

$$K_0 = \begin{bmatrix} \frac{a_3-2a_1+(a_1-a_2)z}{a_1(z-1)} & 0 & 0 \\ 0 & \frac{a_3-2a_2+(a_2-a_1)z}{a_2(z-1)} & 0 \\ 0 & 0 & \frac{a_3[(a_3-1)+(a_1+a_2-2a_3+1)z]}{(a_3-a_2)(a_1-a_3)z} \end{bmatrix}$$

and

$$T_0 = \begin{bmatrix} \frac{a_1-a_3}{a_1(z-1)} & 0 & 0 \\ 0 & \frac{a_2-a_3}{a_2(z-1)} & 0 \\ 0 & 0 & \frac{a_3(a_3-1)(z-1)}{(a_3-a_2)(a_1-a_3)z} \end{bmatrix}$$

Proof 2. Applying the operators $\mathcal{A}_1^{n-2}$, $\mathcal{A}_2^{n-2}$ and $\mathcal{A}_3^{n-2}$ on the identities (19)–(21) respectively, one has

$$\mathcal{A}_1^n = \frac{a_3+(a_1+n-a_2-1)z-2(a_1+n-1)}{(a_1+n-1)(z-1)}\mathcal{A}_1^{n-1} + \frac{a_1+n-1-a_3}{(a_1+n-1)(z-1)}\mathcal{A}_1^{n-2}$$

$$\mathcal{A}_2^n = \frac{a_3+(a_2+n-a_1-1)z-2(a_1+n-1)}{(a_2+n-1)(z-1)}\mathcal{A}_2^{n-1} + \frac{a_2+n-1-a_3}{(a_2+n-1)(z-1)}\mathcal{A}_2^{n-2}$$

$$\mathcal{A}_3^n = \frac{(a_3+n-1)[(a_3+n-2)+(a_1+a_2-2a_3-2n+3)z]}{(a_3+n-1-a_2)(a_1-a_3-n+1)z}\mathcal{A}_3^{n-1} + \frac{(a_3+n-1)(a_3+n-2)(z-1)}{(a_3+n-1-a_2)(a_1-a_3-n+1)z}\mathcal{A}_3^{n-1}$$

or, in matrix form

$$\begin{bmatrix}\mathcal{A}_1^n\\ \mathcal{A}_2^n\\ \mathcal{A}_3^n\end{bmatrix}=\begin{bmatrix}\frac{a_3+(a_1+n-a_2-1)z-2(a_1+n-1)}{(a_1+n-1)(z-1)} & 0 & 0\\ 0 & \frac{a_3+(a_2+n-a_1-1)z-2(a_2+n-1)}{(a_2+n-1)(z-1)} & 0\\ 0 & 0 & \frac{(a_3+n-1)[(a_3+n-2)+(a_1+a_2-2a_3-2n+3)z]}{(a_3+n-1-a_2)(a_1-a_3-n+1)z}\end{bmatrix}$$

$$\times\begin{bmatrix}\mathcal{A}_1^{n-1}\\ \mathcal{A}_2^{n-1}\\ \mathcal{A}_3^{n-1}\end{bmatrix}+\begin{bmatrix}\frac{a_1+n-1-a_3}{(a_1+n-1)(z-1)} & 0 & 0\\ 0 & \frac{a_2+n-1-a_3}{(a_2+n-1)(z-1)} & 0\\ 0 & 0 & \frac{(a_3+n-1)(a_3+n-2)(z-1)}{(a_3+n-1-a_2)(a_1-a_3-n+1)z}\end{bmatrix}\begin{bmatrix}\mathcal{A}_1^{n-2}\\ \mathcal{A}_2^{n-2}\\ \mathcal{A}_3^{n-2}\end{bmatrix}$$

that is,

$$X_n=\mathcal{D}_{n-1}\left(\begin{bmatrix}\frac{a_3-2a_1+(a_1-a_2)z}{a_1(z-1)} & 0 & 0\\ 0 & \frac{a_3-2a_2+(a_2-a_1)z}{a_2(z-1)} & 0\\ 0 & 0 & \frac{a_3[(a_3-1)+(a_1+a_2-2a_3-2n+1)z]}{(a_3-a_2)(a_1-a_3)z}\end{bmatrix}\right)X_{n-1}$$

$$+\mathcal{D}_{n-1}\left(\begin{bmatrix}\frac{a_1-a_3}{a_1(z-1)} & 0 & 0\\ 0 & \frac{a_2-a_3}{a_2(z-1)} & 0\\ 0 & 0 & \frac{a_3(a_3-1)(z-1)}{(a_3-a_2)(a_1-a_3)z}\end{bmatrix}\right)X_{n-2}$$

which may be rewritten in the form

$$X_n=\mathcal{D}_{n-1}(K_0)X_{n-1}+\mathcal{D}_{n-1}(T_0)X_{n-2}$$

or

$$X_n=K_{n-1}X_{n-1}+T_{n-1}X_{n-2} \tag{25}$$

which is the proof of the lemma. □

Now, in order to have our next lemma, let us re-formulate the previous results in matrix form. Use of identities (11) and (12) yields the following matrix equation:

$$\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_2\\ \mathcal{A}_3\end{bmatrix}=\begin{bmatrix}1\\ \frac{a_1}{a_2}\\ \frac{a_1a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}\end{bmatrix}\mathcal{A}_1+\begin{bmatrix}0\\ \frac{a_2-a_1}{a_2}\\ \frac{a_3[a_1-(a_3-a_2)z]}{(a_1-a_3)(a_3-a_2)z}\end{bmatrix}\mathcal{I}:=L_1\mathcal{A}_1+M_1\mathcal{I}$$

also, from 19, 20 and 22

$$\begin{bmatrix}\mathcal{A}_1^2\\ \mathcal{A}_2^2\\ \mathcal{A}_3^2\end{bmatrix}=\begin{bmatrix}\frac{a_3+(a_1+1-a_2)z-2(a_1+1)}{(a_1+1)(z-1)}\\ \frac{a_1[(a_3-2a_2-2)+(a_2-a_1+1)z]}{a_2(a_2+1)(z-1)}\\ \frac{a_1a_3(z-1)(a_3+1)[a_3+(a_1+a_2-2a_3-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2}\end{bmatrix}\mathcal{A}_1$$

$$+\begin{bmatrix}\frac{a_1-a_3+1}{(a_1+1)(z-1)}\\ \frac{(a_2-a_1)[(a_3-2a_2-2)+(a_2-a_1+1)z]+a_2(a_2-a_3+1)}{a_2(a_2+1)(z-1)}\\ \frac{a_3(a_3+1)[[a_1-(a_3-a_2)z][a_3-(2a_3-a_1-a_2+1)z]+(a_1-a_3)(a_3-a_2)(z-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2}\end{bmatrix}$$

$$\mathcal{I}:=L_2\mathcal{A}_1+M_2\mathcal{I} \tag{26}$$

from which

$$X_n=L_n\mathcal{A}_1+M_n\mathcal{I},\quad n=0,1,2 \tag{27}$$

where

$$X_n=\begin{bmatrix}\mathcal{A}_1^n\\ \mathcal{A}_2^n\\ \mathcal{A}_3^n\end{bmatrix} \tag{28}$$

and L_n, M_n, $n = 0, 1, 2$ are the three-dimensional vectors given by

$$L_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad L_1=\begin{bmatrix}1\\ \frac{a_1}{a_2}\\ \frac{a_1a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}\end{bmatrix},$$

$$L_2=\begin{bmatrix}\frac{a_3+(a_1+1-a_2)z-2(a_1+1)}{(a_1+1)(z-1)}\\ \frac{a_1[(a_3-2a_2-2)+(a_2-a_1+1)z]}{a_2(a_2+1)(z-1)}\\ \frac{a_1a_3(z-1)(a_3+1)[a_3+(a_1+a_2-2a_3-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2}\end{bmatrix}$$

and

$$M_0=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad M_1=\begin{bmatrix}0\\ \frac{a_2-a_1}{a_2}\\ \frac{a_3[a_1-(a_3-a_2)z]}{(a_1-a_3)(a_3-a_2)z}\end{bmatrix},$$

$$M_2=\begin{bmatrix}\frac{a_1-a_3+1}{(a_1+1)(z-1)}\\ \frac{(a_2-a_1)[(a_3-2a_2-2)+(a_2-a_1+1)z]+a_2(a_2-a_3+1)}{a_2(a_2+1)(z-1)}\\ \frac{a_3(a_3+1)[[a_1-(a_3-a_2)z][a_3-(2a_3-a_1-a_2+1)z]+(a_1-a_3)(a_3-a_2)(z-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2}\end{bmatrix} \tag{30}$$

Next lemma will deal with the nth power, $n\in\mathbb{Z}$, of any shifted operators $\mathcal{A}_i$, $i=1,2,3$, as a recurrence relation of the operator $\mathcal{A}_1$ and $\mathcal{I}$.

Lemma 7. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the operators defined as in Definition (1), then*

$$X_n=L_n\mathcal{A}_1+M_n\mathcal{I};\qquad \forall n\in\mathbb{Z} \tag{31}$$

where X_n defined as in (28), and $\forall n\in\mathbb{Z}$, we have

$$\left.\begin{aligned}L_n&=K_{n-1}L_{n-1}+T_{n-1}L_{n-2},\\ M_n&=K_{n-1}M_{n-1}+T_{n-1}M_{n-2}\end{aligned}\right\},\quad n\geqslant 0 \tag{32}$$

and

$$\left.\begin{aligned}L_n&=T_{n+1}^{-1}[L_{n+2}-K_{n+1}L_{n+1}],\\ M_n&=T_{n+1}^{-1}[M_{n+2}-K_{n+1}M_{n+1}]\end{aligned}\right\},\quad n<0 \tag{33}$$

Moreover, L_n and M_n for n = 0, 1, 2 are defined as in (29) and (30) and K_n, T_n are defined as in Lemma (6)

Proof 3. Results obtained in example (3), gives the proof of the lemma when n = 0,1,2. Using mathematical induction, assume that

$$X_k=L_k\mathcal{A}_1+M_k\mathcal{I}\quad\text{and}\quad X_{k-1}=L_{k-1}\mathcal{A}_1+M_{k-1}\mathcal{I},\qquad k=1,2,\ldots$$

then from Lemma (6), we have

$$\begin{aligned}X_{k+1}&=K_kX_k+T_kX_{k-1}\\ &=K_k(L_k\mathcal{A}_1+M_k\mathcal{I})+T_k(L_{k-1}\mathcal{A}_1+M_{k-1}\mathcal{I})\\ &=(K_kL_k+T_kL_{k-1})\mathcal{A}_1+(K_kM_k+T_kM_{k-1})\mathcal{I}\\ &=L_{k+1}\mathcal{A}_1+M_{k+1}\mathcal{I}\end{aligned}$$

from which it follows that

$$X_n=L_n\mathcal{A}_1+M_n\mathcal{I},\qquad \forall n\in\mathbb{Z}$$

where

$$\left.\begin{array}{l} L_{k+1} = K_k L_k + T_k + L_{k-1}, \\ M_{k+1} = K_k M_k + T_k + M_{k-1} \end{array}\right\} \tag{34}$$

which completes the proof of the lemma. □

One can easily show that (34), can be split into positive and negative cases as in (32 and 33).

The previous two lemmas asserts the existence of a unique representation for $\mathcal{A}_i^\alpha$, $\mathcal{A}_j^\beta$; i, $j = 1,2,3$ and $\alpha, \beta \in \mathbb{Z}$ in terms of the operators $\mathcal{A}_1$ and $\mathcal{I}$ with coefficients $l_{i\alpha}$, $m_{i\alpha}$ and $l_{j\beta}$, $m_{j\beta}$ which are the ith and the jth elements of the vectors L_α, M_α and L_β, M_β respectively,

$$\mathcal{A}_i^j = l_{ij}\mathcal{A}_1 + m_{ij}\mathcal{I}, \qquad i = 1,2,3; \quad j \in \mathbb{Z} \tag{35}$$

Consequently, the recurrence relation combining $\mathcal{A}_i^\alpha$ and $\mathcal{A}_j^\beta$ is unique.

The technique of our work depends essentially on the relations between the shifted operators defined in (8)–(12) on Gauss functions which enable us to find the desired formulas. The following theorem gives a general form of the relation between the three Gauss functions:

$$_2F_1[a_1+\alpha_1, a_2; a_3; z], \quad {_2F_1}[a_1 + \alpha_2, a_2; a_3; z] \quad \text{and} \quad {_2F_1}[a_1+\alpha_3, a_2; a_3; z] \tag{36}$$

Theorem 8. *Let $\alpha_i \in \mathbb{Z}, i = 1,2,3$, then the three Gauss functions (43) are related in the following linear form:*

$$\begin{vmatrix} \mathcal{A}_1^{\alpha_1} & \mathcal{A}_1^{\alpha_2} & \mathcal{A}_1^{\alpha_3} \\ l_1\alpha_1 & l_1\alpha_2 & l_1\alpha_3 \\ m_1\alpha_1 & m_1\alpha_2 & m_1\alpha_3 \end{vmatrix} {_2F_1}[a_1, a_2; a_3; z] = 0 \tag{37}$$

where $A_1^{\alpha_i}$, $i = 1,2,3$ are defined as in Definition (1), and l_{ij}, m_{ij} are the ith elements of the vectors L_i, M_j respectively.

Proof 4. Assuming that the desired recurrence relation joining the three mentioned Gauss function is

$$d_1 {_2F_1}[a_1+\alpha_1, a_2; a_3; z] + d_2 {_2F_1}[a_1+\alpha_2, a_2; a_3; z] + d_3 {_2F_1}[a_1 + \alpha_3, a_2; a_3; z] = 0$$

for some nonzero $d_i \in \mathbb{Z}$, which can be written in operator form as

$$\sum_{k=1}^{3} (d_k \mathcal{A}_1^{\alpha_k}) {_2F_1}[a_1, a_2; a_3; z] = 0 \tag{38}$$

Now, setting $i = 1$ in (35), we get

$$\mathcal{A}_1^j = l_{1j}\mathcal{A}_1 + m_{1j}\mathcal{I}, \quad j \in \mathbb{Z}$$

and for $j = \alpha_k$, $k = 1,2,3$, we get

$$\mathcal{A}_1^{\alpha_k} = l_{1\alpha_k}\mathcal{A}_1 + m_{1\alpha_k}\mathcal{I}, \quad k = 1,2,3 \tag{39}$$

combining (38) and (39), we get

$$\left.\begin{array}{l} \sum_{k=1}^{3} d_k l_{1\alpha_k} = 0, \quad \text{and} \\ \sum_{k=1}^{3} d_k m_{1\alpha_k} = 0 \end{array}\right\} \tag{40}$$

solving for d_k, one can have

$$d_k = \begin{vmatrix} l_{1\alpha_{k+1}} & l_{1\alpha_{k+2}} \\ m_{1\alpha_{k+1}} & m_{1\alpha_{k+2}} \end{vmatrix}, \quad k = 1,2,3,\ldots (\text{mod } 3) \tag{41}$$

substituting in (38) we get

$$\sum_{k=1}^{3} \left(\begin{vmatrix} l_{1\alpha_{k+1}} & l_{1\alpha_{k+2}} \\ m_{1\alpha_{k+1}} & m_{1\alpha_{k+2}} \end{vmatrix} \mathcal{A}_1^{\alpha_k} \right) {_2F_1}[a_1, a_2; a_3; z] = 0$$

which can be written in the determinant form as in (38), which completes the proof. □

A similar formulas for the shifts in a_2 and a_3 can be easily obtained. The next corollary generalizes the result obtained in Theorem (8).

Corollary 9. *Let $\mathcal{A}_i^{\alpha_k}$, i, $k = 1,2,3$ be the shifted operator defined as in Definition (1), then the recurrence relation joining the three Gauss functions with one parameter shifted is given by:*

$$\begin{vmatrix} \mathcal{A}_i^{\alpha_1} & \mathcal{A}_i^{\alpha_2} & \mathcal{A}_i^{\alpha_3} \\ l_i\alpha_1 & l_i\alpha_2 & l_i\alpha_3 \\ m_i\alpha_1 & m_i\alpha_2 & m_i\alpha_3 \end{vmatrix} {_2F_1}[a_1, a_2; a_3; z] = 0, \quad i = 1,2,3 \tag{42}$$

The following theorem deals with the more general formula joining the three shifted Gauss polynomials:

$$_2F_1[a_1+\alpha_1, a_2; a_3; z], \ {_2F_1}[a_1, a_2 + \alpha_2; a_3; z] \quad \text{and} \ {_2F_1}[a_1, a_2; a_3+\alpha_3; z] \tag{43}$$

Theorem 10. *For any integers α_i, $i = 1, 2, 3$, the three shifted Gauss polynomials (43), are linearly related in the form:*

$$\begin{vmatrix} \mathcal{A}_1^{\alpha_1} & \mathcal{A}_2^{\alpha_2} & \mathcal{A}_3^{\alpha_3} \\ l_1\alpha_1 & l_2\alpha_2 & l_2\alpha_3 \\ m_1\alpha_1 & m_2\alpha_2 & m_2\alpha_3 \end{vmatrix} {_2F_1}[a_1, a_2; a_3; z] = 0 \tag{44}$$

where l_{ij}, m_{ij}; $i, j = 1, 2, 3$ are defined as in (32 and 33).

Proof 5. Suppose that the required relation has the form:

$$d_1 \ {_2F_1}[a_1+\alpha_1, a_2; a_3; z] + d_2 \ {_2F_1}[a_1, a_2+\alpha_2; a_3; z] + d_3 \ {_2F_1}[a_1, a_2; a_3+\alpha_3; z] = 0$$

for some $d_i = d_i(a_1, a_2, a_3, z)$, $i = 1, 2, 3$, that is

$$\sum_{i=1}^{3} (d_i \mathcal{A}_i^{\alpha_i}) \ {_2F_1}[a_1, a_2; a_3; z] = 0 \tag{45}$$

Now, setting $j = \alpha_i \in \mathbb{Z}$, $i = 1,2,3$ in (35), one can have

$$\mathcal{A}_i^{\alpha_i} = l_{i\alpha_i}\mathcal{A}_1 + m_{i\alpha_i}\mathcal{I}, \quad i = 1,2,3 \tag{46}$$

substituting in (45)

$$\left.\begin{array}{l} \sum_{i=1}^{3} d_i l_{i\alpha_i} = 0, \ \text{and} \\ \sum_{i=1}^{3} d_i m_{i\alpha_i} = 0 \end{array}\right\} \tag{47}$$

solving for d_i, we get

$$d_i = \begin{vmatrix} l_{i+1,\alpha_{i+1}} & l_{i+2,\alpha_{i+2}} \\ m_{i+1,\alpha_{i+1}} & m_{i+2,\alpha_{i+2}} \end{vmatrix}, i = 1, 2, 3, \ldots (\text{mod } 3) \tag{48}$$

using (48), we can rewrite (45) in the form (44) which completes the proof. □

4. Computational examples

In this section we will use the computer algebra system, *Mathematica* to find the vectors L_j, M_j for all integer j, and consequently expressing the operators $\mathcal{A}_i^j$, $i = 1, 2, 3$ for all values of j, in terms of the operators $\mathcal{A}_1$, $\mathcal{I}$.

Referring to Lemma (6) in the previous section, one can easily derive the relation

$$\mathcal{A}_i^j = l_{ij}\mathcal{A}_1 + m_{ij}\mathcal{I}, \quad i = 1, 2, 3 \text{ and } j \in \mathbb{Z}$$

where l_{ij},m_{ij} are the ith elements of the vectors L_j and M_j.

The following tables give the values of l_{ij}, m_{ij} for $i = 1, 2, 3$ and $j = 0, 1, 2$.

	j = 0	**j** = 1	**j** = 2
i = 1	$l_{10} = 0$	$l_{11} = 1$	$l_{12} = \frac{a_3+(a_1-a_2+1)z-2(a_1+1)}{(a_1+1)(z-1)}$
i = 2	$l_{20} = 0$	$l_{21} = \frac{a_1}{a_2}$	$l_{22} = \frac{a_1[(a_3-2a_2-2)+(a_2-a_1+1)z]}{a_2(a_2+1)(z-1)}$
i = 3	$l_{30} = 0$	$l_{31} = \frac{a_1a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}$	$l_{32} = \frac{a_1a_3(a_3+1)(z-1)[a_3-(2a_3-a_1-a_2+1)z]}{(a_1-a_3)(a_1-a_3-1)(a_3-a_2)(a_3-a_2+1)z^2}$

and

	j = 0	**j** = 1	**j** = 2
i = 1	$m_{10} = 1$	$m_{11} = 0$	$m_{12} = \frac{a_1-a_3+1}{(a_1+1)(z-1)}$
i = 2	$m_{20} = 1$	$m_{21} = \frac{a_2-a_1}{a_2}$	$m_{22} = \frac{(a_2-a_1)[(a_3-2a_2-2)+(a_2-a_1+1)z]+a_2(a_2-a_3+1)}{a_2(a_2+1)(z-1)}$
i = 3	$m_{30} = 1$	$m_{31} = \frac{a_3[a_1-(a_3-a_2)z]}{(a_1-a_3)(a_3-a_2)z}$	$m_{32} = \frac{a_3(a_3+1)[(a_1-(a_3-a_2)z)(a_3-(2a_3-a_1-a_2+1)z)+(a_1-a_3)(a_3-a_2)(z-1)z]}{(a_1-a_3)(a_1-a_3-1)(a_3-a_2)(a_3-a_2+1)z^2}$

Using the relations

$$\mathcal{A}_1^2 = l_{12}\mathcal{A}_1 + m_{12}\mathcal{I} \tag{49}$$

$$\mathcal{A}_2^2 = l_{22}\mathcal{A}_1 + m_{22}\mathcal{I} \tag{50}$$

and

$$\mathcal{A}_3^2 = l_{32}\mathcal{A}_1 + m_{32}\mathcal{I} \tag{51}$$

we can easily obtain $\mathcal{A}_1^2$, $\mathcal{A}_2^2$ and $\mathcal{A}_3^2$ which are exactly what we obtained in example (4).

Moreover, one can easily use the previous computations to present a recurrence relation joining any two operators such as $\mathcal{A}_2^2$ and $\mathcal{A}_3^2$, by eliminating $\mathcal{A}_1$ from Eqs. (50) and (51), we get

$$l_{32}\mathcal{A}_2^2 - l_{22}\mathcal{A}_3^2 = (l_{32}m_{22} - l_{22}m_{32})\mathcal{I}$$

which may be rewritten as

$$l_{32}\ {}_2F_1[a_1, a_2 + 2; a_3; z] - l_{22}\ {}_2F_1[a_1, a_2; a_3 + 2; z] \\ - (l_{32}m_{22} - l_{22}m_{32})_2F_1[a_1, a_2; a_3; z] = 0$$

where l_{22}, l_{32}, m_{22} and m_{32} can be obtained from the above tables.

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